Defects localization applied to the inverse medium problem

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Abstract

We investigate numerical methods to retrieve a piece-wise constant approximation of an acoustic refraction index from far-field measurements.

We here propose to enhance this reconstruction by coupling it, in two different strategies, with a previously developed defects localization method. Both strategies can be combined and are aimed to reducing the number of computed parameters.

Moreover, our defects localization provides a new (constructive) characterization of an unknown refraction index. We thus investigate the minimization of defects as a new approach to solve the inverse medium problem. Our results are illustrated by numerical experiments.

1 Introduction

In inverse acoustic scattering, one tries to recover information about a scatterer from measurements. The penetrable scatterers we are interested in are also called inhomogeneous media and are characterized by a refraction index $n \in L^\infty(\mathbb{R}^d)$, where $d = 2$ or 3 [1]. We place ourselves in the case of $(n-1)$ having a compact support.

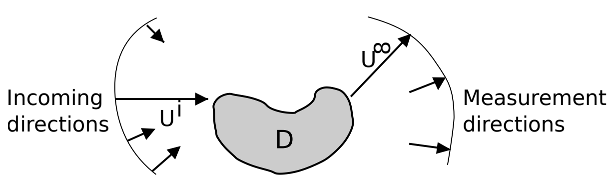


Figure 1: : General setting and notations.

1.1 The direct problem

The acoustic total field $u_n \in L^2_{loc}(\mathbb{R}^d)$ is assumed to satisfy the Helmholtz equation

$$\Delta u_n + k^2 n(x) u_n = 0, \quad x \in \mathbb{R}^d.$$

For practical reasons, we consider plane-wave sources. Hence, the corresponding total field is parameterized by the incidence direction taken in S^{d-1} (see Figure 1). Finally, $u_n^\infty \in \mathcal{C}^\infty(S^{d-1} \times S^{d-1})$ is the associated far-field pattern [1] and $\mathcal{F} : n \mapsto u_n^\infty$ denotes the index-to-far-field mapping.

1.2 The inverse medium problem

With $D = \cup Z_i$, $i = 1 \dots N$, we look for a piece-wise constant approximation $n(x) = \sum \eta_i \mathbf{1}_{Z_i}(x)$ of the actual refraction index, denoted by $n^\star \in L^\infty(\mathbb{R}^d)$, from the corresponding far-field measurements $u_{n^\star}^\infty$. A popular method to approximate $n^\star$, for its ease of implementation and efficiency, is using the iterative Gauss-Newton (G.-N.) method to minimize the following regularized cost function [2]

$$J(n) := \|\mathcal{F}(n) - u_{n^\star}^\infty\|^2_{L^2(S^{d-1}\times S^{d-1})} + \lambda \|n - n_0\|^2_{L^2(D)}.$$

2 Enhancement of piece-wise constant reconstructions through selective focusing

The G.-N. method involves heavy computations in which all parameters η_i are updated at each iteration. However, the initial guess could be exact in some zones Z_i and thus, the corresponding constants should not be updated. Also, during the reconstruction, some constants can reach a satisfactory precision while the other ones still require improvement.

2.1 Defects localization

To address these aspects of the reconstruction, the useful information would thus be a fast localization of the exact (enough) constants. To this end, we have extended the so-called Factorization method (see [3] and references therein) to localize the differences between $n^\star$ and a fixed (known) reference index. We call these differences defects and their localization is achieved *via* a localization function: for each $x \in \mathbb{R}^d$, we have the equivalence between $n(x) \neq n^\star(x)$ and

$$\mathcal{S}_{\{n,n^\star\}}(x) := \left(\sum_j \frac{\left| \langle \overline{u_n(\cdot, x)}, \psi_j \rangle_{L^2(S^{d-1})} \right|^2}{\sigma_j} \right)^{-1} > 0,$$

where (σ_j, ψ_j) is an eigen-system of the self-adjoint operator $W_\# := |W + W^\star| + |W - W^\star|$, where

$$W := (id + \alpha F_n)^\star (F_{n^\star} - F_n),$$

F_n is the classical far-field operator defined by

$$F_n g(\hat{x}) := \langle g, \overline{u_n^\infty(\cdot, \hat{x})} \rangle_{L^2(S^{d-1})},$$

and α is a constant. So, $\mathcal{S}_{\{n,n^\star\}}$ is built only from the measurements $u_{n^\star}^\infty$ and the reference index n.

2.2 *Selective reconstruction*

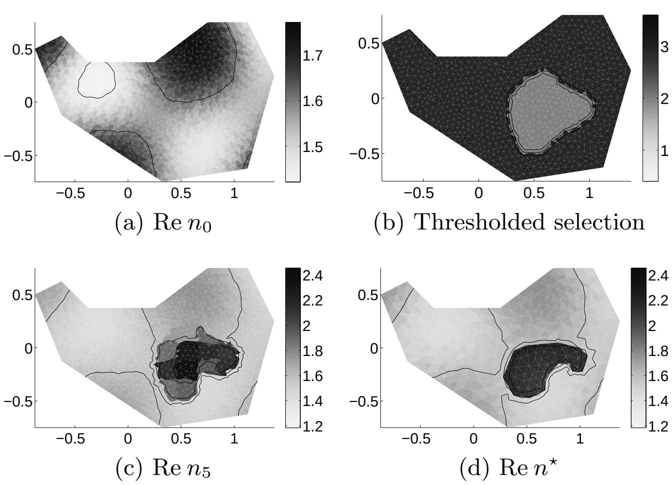
First, we consider the case where $n^\star$ is a locally perturbed version of a known initial state, denoted by n_0. These perturbations can now be localized through the function $\mathcal{S}_{\{n_0, n^\star\}}$. So, only the corresponding constants need to be reconstructed, using n_0 as an initial guess. This naturally provides a substantial reduction in computational costs.

2.3 *Adaptive refinement*

Secondly, we propose an iterative refinement strategy for the reconstruction: starting with $p = 0$,

1. Compute the average value of $\mathcal{S}_{\{n_p, n^\star\}}$ over each zone Z_i.
2. Split the zone corresponding to the highest average value into four and duplicate the corresponding parameter accordingly.
3. Run the G.-N. method on this new set of parameters to compute the approximation n_{p+1}.
4. $p \leftarrow p + 1$ and go to 1.

This leads to an approximation of $n^\star$ with a constrained number of parameters, positioned to fit as much as possible the geometry of this index.



(a) Re n_0 (b) Thresholded selection (c) Re n_5 (d) Re $n^\star$

Figure 2: : Full reconstruction with 16 parameters.

2.4 *Combination of both strategies*

Both strategies can be chained. Figure 2a depicts the real part of some unperturbed index n_0. The perturbation is then located by thresholding the values of $\mathcal{S}_{\{n_0, n^\star\}}$ (Figure 2b). Lastly, five iterations of the adaptive refinement process are applied to this selection, starting with a single parameter and ending with only 16. The result is shown on Figure 2c and can be compared to the actual index $n^\star$ on Figure 2d. The final relative error is $\|n_5 - n^\star\| \,/\, \|n^\star\| = 0.07$.

3 A new approach to the inverse medium problem

Lastly, the construction of $\mathcal{S}_{\{n, n^\star\}}$ provides a new constructive uniqueness proof for the inverse medium problem that is valid in $\mathbb{R}^3$, but also in $\mathbb{R}^2$, and for any k. Indeed, if $u_n^\infty = u_{n^\star}^\infty$, then $\mathcal{S}_{\{n, n^\star\}} = 0$ and thus, $n = n^\star$. Therefore, we propose a new way to look for $n^\star$ by minimizing

$$J_{\mathcal{S}}(n) := \left\|\mathcal{S}_{\{n, n^\star\}}\right\|^2_{L^2(D)} + \lambda \left\|n - n_0\right\|^2_{L^2(D)}.$$

This approach shows encouraging numerical results when compared to the classical cost function J. Also, since the localization function is defined locally, its minimization on any sub-part of D should allow the reconstruction of the unknown index $n^\star$ on this subpart. Thus, in theory, this new method handles domain decomposition straightforwardly, although we have no numerical evidence at this point.

Conclusion and perspectives

The inverse medium problem's numerical resolution has been enhanced in two specific cases by coupling it with a defects localization method. Moreover, this defects localization provides a new reconstruction approach that shows promising results.

Further investigations are performed to extend the localization function and to establish its regularity. That information is needed to develop, in particular, domain decomposition and L^1-norm minimization for our new approach to the inverse medium problem.

References

[1] D. Colton, J. Coyle, and P. Monk. Recent developments in inverse acoustic scattering theory. *SIAM Rev.*, 42(3):369–414 (electronic), 2000.

[2] H.W. Engl, M. Hanke, and A. Neubauer. *Regularization of inverse problems*, volume 375. Springer Netherlands, 1996.

[3] Y. Grisel, V. Mouysset, P-A. Mazet, and J-P. Raymond. Determining the shape of defects in non-absorbing inhomogeneous media from far-field measurements. *Inverse Problems*, 28:055003, 2012.