An Application of the Factorization Method to the Detection of Inclusions in Acoustics

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Abstract

The factorization method reconstructs a scatterer's domain. This is done by characterizing the range of one of the factors in an appropriated factorization of the far field operator. Knowing the far field data of a scatterer with (experimentally) and without (numerically) an inhomogeneity, this method has been extended to localize inclusions. We present here some numerical results and a brief explanation of this extension.

Introduction

We are interested in retrieving the domain $D \in \mathbb{R}^d$ (d=2 or 3) of a scatterer with index of refraction $n(x)$. Thus we consider the Helmholtz equation for the scattered field u^s and an index of refraction $n(x)$ modelling the diffraction of D by an incident wave h

$$\begin{cases} \Delta u^s + k^2 n(x) u^s = -k^2 n(x) h, \\ \text{Sommerfeld radiation condition on } u^s. \end{cases} \quad (1)$$

In the past two decades some numerical methods for fast detection have been developed, as the Linear Sampling Method by Colton *et al.* (see [1] and references) or the Factorization Method by Kirsch [3]. This last one consists in a constrained optimization problem: $\forall z \in \mathbb{R}^d$, $z \in D \iff$

$$\inf \left\{ \left| \langle F\psi, \psi \rangle_{L^2(\Gamma)} \right| , \langle \Phi_z^\infty, \psi \rangle_{L^2(\Gamma)} = 1 \right\} > 0, \quad (2)$$

F being the far field operator, Γ a part of the unit sphere S_1^d and Φ_z^∞ the far field of Green's function for (1) with $n(x) = 1$ (free space). This method extends to treatment of incomplete data or penetrable objects. It takes its name from the use of a symmetric factorization of the far field operator : $F = GAG^\star$. In a first step it is shown that $z \in D \iff \Phi_z^\infty \in \mathcal{R}(G)$, and then that $\Phi_z^\infty \in \mathcal{R}(G)$ is equivalent to (2).

Gebauer [2] has worked on the reconstruction of an inhomogeneity with domain Ω included in D by the knowledge of the Neumann-to-Dirichlet maps in D with and without the inhomogeneity (modeled respectively by indexes of refraction n_1 and n_0), respectively named Λ_1 and Λ_0. The factorization is then $\Lambda_1 - \Lambda_0 = LFL^\star$ and their ranges are connected by the relation $\mathcal{R}\left((\Lambda_1 - \Lambda_0)^{\frac{1}{2}}\right) = \mathcal{R}(L)$. The point of this factorization is that $z \in \Omega \iff v_z \in \mathcal{R}(L)$, $\{v_z\}_z$ being an suitable family of functions. Main problem is that constructing this family requires the knowledge of Green's function for index of refraction $n_0(x)$.

In this paper we will present a method localizing inclusions with the single knowledge of far fields obtained with n_0 and n_1. Some numerical results will show its efficiency.

1 Construction of the method

We studied a similar configuration but in the far field case. To avoid the request of the fundamental solution to (1) with $n = n_0(x)$ we tried to reconstruct Ω with the single knowledge of far field datas for n_1 and for n_0. If u_1 is solution to (1) with $n = n_1(x)$ (the "real measurements") and u_0 is solution to (1) with $n = n_0(x)$ (the scatterer without the inclusion, obtained by computation), we note their respective far fields u_1^∞ and u_0^∞ (see figure 1).

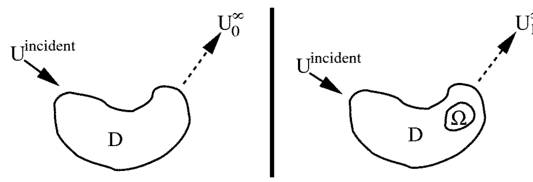

Figure 1: Sketch of geometry.

Let then F_1 and F_0 be the associated far field operators mapping a distribution $h \in L^2(S_1^d)$ of incoming plane waves to the resulting far field : $F_j : h \mapsto u_j^\infty$ ($j = 0$ or 1). Showing that it permits to recover Ω, we will use the algorithm presented in [3] with

$$\tilde{F} := F_1 - F_0.$$

2 Mathematical justification

There are two problems to adapt the original result to $\tilde{F}$. First, as pointed out before, the natural family of functions suited to characterize Ω requires the fundamental solution to (1) with $n = n_0(x)$ while we only assume u_0^∞ to be known. Secondly $F_1 - F_0$

yields to a non symmetric factorization. For the sake of simplicity let us design by : $T_j h := \int_{\mathbb{R}^d} k^2 m_j h \Phi_z$, $T_j^\infty h := \int_{\mathbb{R}^d} k^2 m_j h \Phi_z^\infty$ ($j = 0$ or 1), Hh is the Herglotz function with kernel $h \in L^2(S_1^d)$ and we introduce the following isomorphisms of $L^2(D)$: $A_1 := I + T_0 + T_1, A_0 := I + T_0$. Then the "pseudo far field operator" $\tilde{F}$ can be written this way:

$$\tilde{F} = (G_A - G_B)H,$$

with $G_A := T_1^\infty A_1^{-1}$ and $G_B := T_0^\infty (A_0^{-1} - A_1^{-1})$.

Characterization of Ω by $\mathcal{R}(G_A)$:

Considering the family of functions $\{\Phi_z^\infty\}_z$ and studying the interior transmission problem, we show

$$z \in \Omega \iff \Phi_z^\infty \in \mathcal{R}(G_A).$$

Link between $\mathcal{R}(G_A)$ and $\tilde{F}$:

Main result is that $\Phi_z^\infty \in \mathcal{R}(G_A)$ if and only if

$$\inf \left\{ \left| \langle \tilde{F}\psi, \psi \rangle_{L^2(\Gamma)} \right|, \text{ with } \langle \Phi_z^\infty, \psi \rangle_{L^2(\Gamma)} = 1 \right\} > 0.$$

This usually comes from a symmetric factorization of $\tilde{F}$ not straightforwardly reachable here. However, noting by "$\star$" the adjoint operators in the weighted L^2-space $L^2(|m_1|, \Omega)$, we have :

$$\tilde{F} = (G_A - G_B) C_{m_1}^\star A_1^\star G_A^\star,$$

where $C_{m_1}^\star h = \frac{1}{\gamma k^2} \frac{\overline{m_1}}{|m_1|} h$ and $\gamma = \frac{e^{i\pi/4}}{\sqrt{8\pi k}}$ if $d = 2$, or $\gamma = \frac{1}{4\pi}$ if $d = 3$. This form shows a symmetric part in G_A but with a perturbation term in G_B. So:

• Assume $\Phi_z^\infty \notin \mathcal{R}(G_A)$. Then there exists a sequence Ψ_n such that $\langle \Phi_z^\infty, \Psi_n \rangle = 1$ and $|A_1^\star G_A^\star \Psi_n| \to 0$ (see [3]). Continuity of operators leads to $\left| \left\langle \Psi_n, \tilde{F}\Psi_n \right\rangle \right|$ being majored by

$$\leq c_1 \left(|A_1^\star G_A^\star \Psi_n| + |G_B^\star \Psi_n| \right) |A_1^\star G_A^\star \Psi_n|.$$

By definition of G_A, $|A_1^\star G_A^\star \Psi_n| = |T_1^{\infty\star} \Psi_n|$ and by definition of G_B we have $|G_B^\star \Psi_n| \leq c_2 |T_0^{\infty\star} \Psi_n|$. Now if we evaluate these adjoints we find $T_1^{\infty\star} = \frac{\overline{m_1}}{|m_1|} k^2 H$ and $T_0^{\infty\star} = \frac{\overline{m_0}}{|m_1|} k^2 H$. This shows that when $|T_1^{\infty\star} \Psi_n|$ tends to 0, then $|T_0^{\infty\star} \Psi_n|$ tends to 0 too, and thus $(|A_1^\star G_A^\star \Psi_n| + |G_B^\star \Psi_n|)$ vanishes.

• Reversely, assume $\Phi_z^\infty \in \mathcal{R}(G_A)$. Then there exists φ_0 such that $G_A A_1 \varphi_0 = \Phi_z^\infty$. Let us take Ψ satisfying $\langle \Psi, \Phi_z^\infty \rangle = 1$. The continuity of operators leads to the following minoration : $\left| \left\langle \Psi, \tilde{F}\Psi \right\rangle \right| = \left| \left\langle (G_A^\star - G_B^\star)\Psi, C_{m_1}^\star A_1^\star G_A^\star \Psi \right\rangle \right|$ $\geq |A_1^\star G_A^\star \Psi|^2 \left(c_3 - c_4 |G_B^\star \Psi| |G_A^\star \Psi|^{-1} \right)$.
Since $|A_1^\star G_A^\star \Psi|^2 |\varphi_0|^2 / |\varphi_0|^2 \geq 1/|\varphi_0|^2$ by assumption on φ_0 and Ψ, this is strictly positive if $|G_B^\star \Psi| |G_A^\star \Psi|^{-1}$ is small enough. This is achieved for $|m_0|$ small in comparison to $|m_1|$.

3 Numerical results

We simulate non symmetric 2-D shapes, n_0 and n_1 being constant or piecewise constant, and Ω being one or more non connected non symmetric shapes. Figure 2 shows a sample reconstruction of an inclusion with 5% noise on the "measured results" with the following parameters : $k = 3$, $n_0 = 1.2$, $n_1 = 1.2$ in $D \setminus \Omega$, $n_1 = 2.2$ in Ω, and $n_1 = n_0 = 1$ out of D.

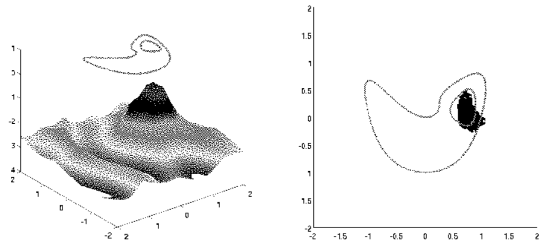

Figure 2: Localization of inclusion.

Conclusion

We have extended the factorization method to locate inclusions by the single knowledge of the far field data of the including object with and without this inclusions with encouraging numerical results. The present demonstration and our numerical experiments show that the localization depends on the contrast between the object and the inclusion. An improvement would be a control over this dependence to achieve a better localization with small contrasts.

References

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