Localization of Defects and Applications to Parameter Identification

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Physical background

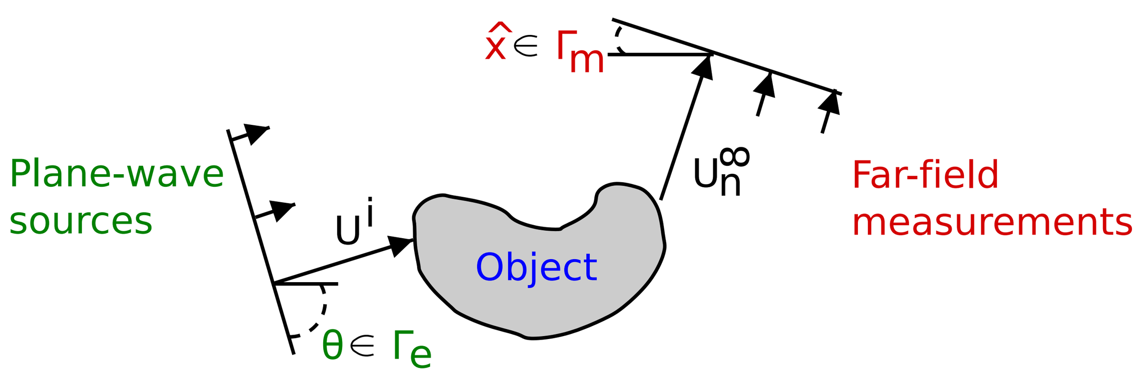


Figure: Acoustic scattering: plane wave incidence directions and far-field measurements.

- Problem : recover information about a scatterer from far field data

Goals

1. Reconstruct the scatterer's refraction index through an iterative numerical method
2. Build a fast numerical method to locate defects in some reference inhomogeneous refraction index.
3. Investigate the coupling of these methods.

Mathematical setting

- Plane wave sources : $u^i(x) := e^{ikx\cdot\vec{\theta}}$, $x \in \mathbb{R}^d$, $\vec{\theta} \in \Gamma_e$
- Helmholtz equation for inhomogeneous media in an unbounded domain:

$$\begin{cases} \Delta u^s + k^2 n(x) u^s = -k^2(n(x)-1)u^i, \ x \in \mathbb{R}^d, \\ \lim\limits_{|x|\to\infty} |x|^{\frac{d-1}{2}}(\partial_{|x|}u^s - iku^s) = 0. \end{cases}$$

- Far-field pattern : $u^s(x) = \frac{e^{ik|x|}}{|x|^{\frac{d-1}{2}}} u^\infty\left(\frac{x}{|x|}\right) + o\left(\frac{1}{|x|^{\frac{d-1}{2}}}\right)$, $x \in \mathbb{R}^d$, $\hat{x} \in \Gamma_m$
- Far-field operator : $Fg(\hat{x}) := \int_{\Gamma_e} u^\infty(\vec{\theta}, \hat{x}) g(\vec{\theta}) d\vec{\theta}$

- Problem: extract some information about the *actual* medium's index $n_\star \in L^\infty(O)$ from far-field measurements $u^\infty_\star \in \mathcal{C}^\infty(\Gamma_e, \Gamma_m)$ and an *inhomogeneous reference* medium's index $n \in L^\infty(O)$.
- Difficulties: non-linear and ill-posed inverse problem

Localization of defects

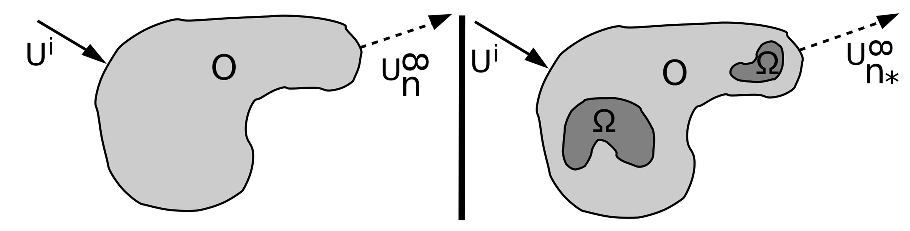


Figure: Reference far-field and actual measurements

Theorem

- $n(x), n_\star(x) \in \mathbb{R}$
- $(n - n_\star)(x) > 0$ or < 0
- Incoming and measurement directions covering the whole unit sphere

$$n(x) \neq n_\star(x) \iff 0 < \mathcal{M}_{\{n,n_\star\}}(x)$$

where W is an operator built from the measurements, and u is the total field for the reference index n.

$$\mathcal{W} := (id + 2ik|\gamma|^2 F)^\star (F_\star - F)$$

$$\mathrm{Re}\, L := (L + L^\star)/2,\ \mathrm{Im}\, L := (L - L^\star)/2i,\ |L| := (L^\star L)^{\frac{1}{2}}$$

$$W := |\mathrm{Re}\,\mathcal{W}| + |\mathrm{Im}\,\mathcal{W}|$$

$$\mathcal{M}_{\{n,n_\star\}}(x) := \|W^{-\frac{1}{2}}\overline{u(\cdot, x)}\|^{-2}_{L^2(\Gamma_e)}$$

Perspective

Conjectured measurement operator $W := |F_\star - F| \implies$ Extension of the localization to limited aperture data and absorbing media

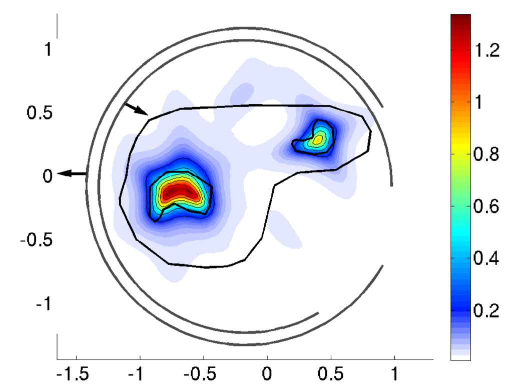
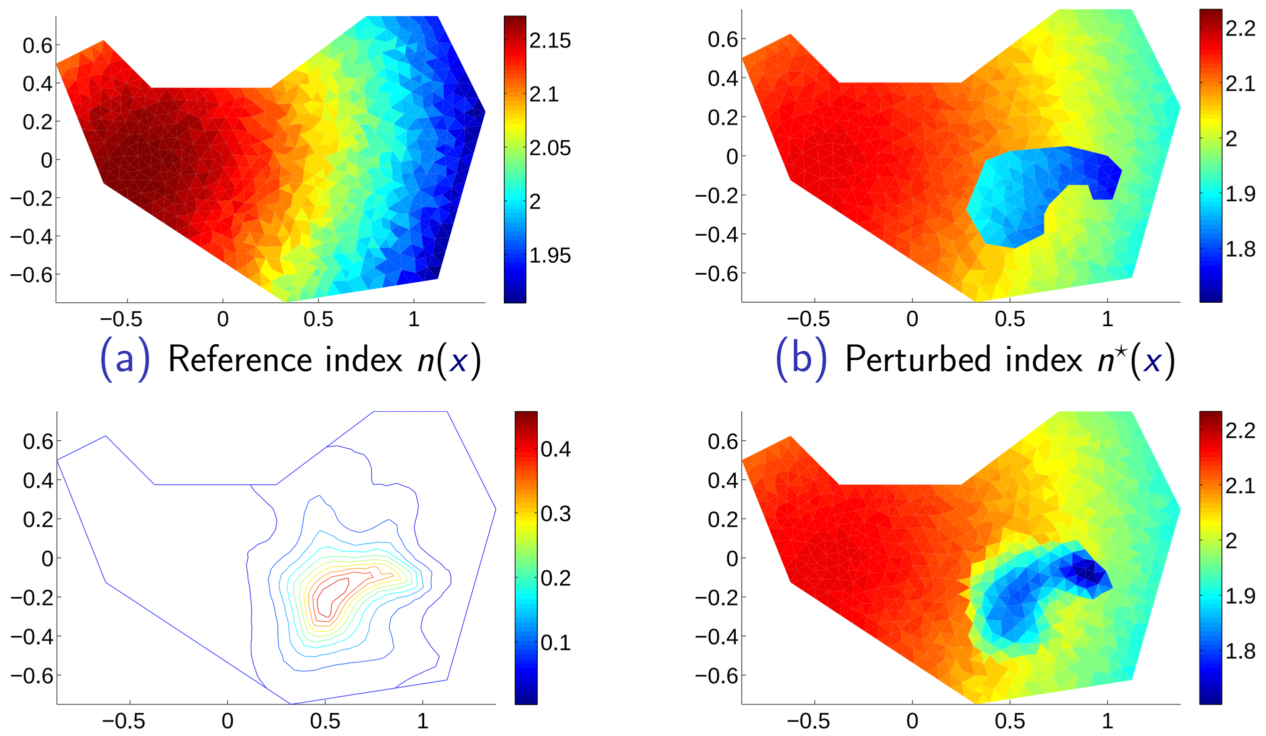


Figure: Plot of $\mathcal{M}_{\{n,n_\star\}}(x)$ for a 2D object with two defects

Perspective

Coupling with the inverse problem of finding u from u^∞ without knowing $n \implies$ Motion detection through successive snapshots $u^\infty_1, u^\infty_2, \ldots$

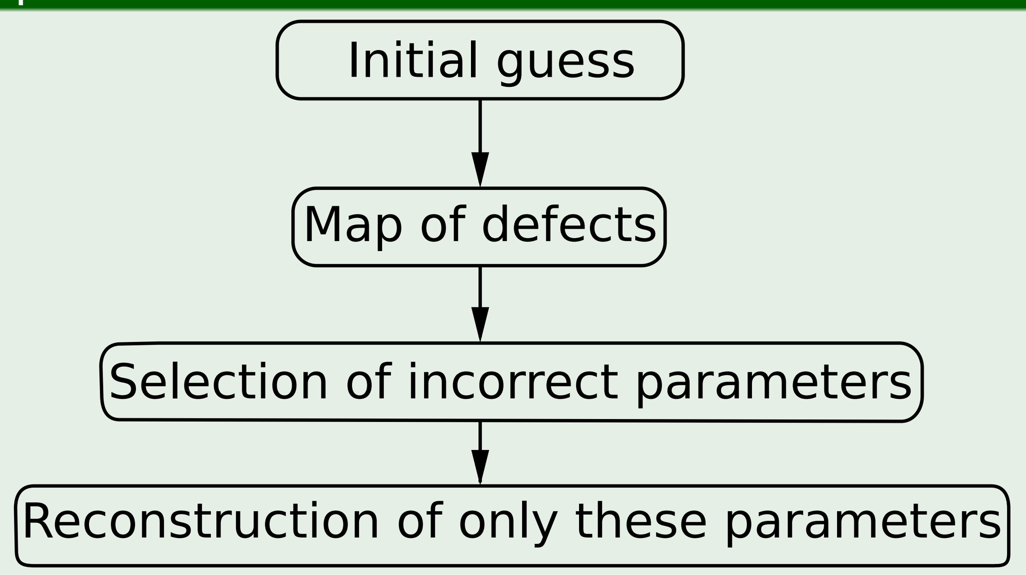
Application 1: selective reconstruction



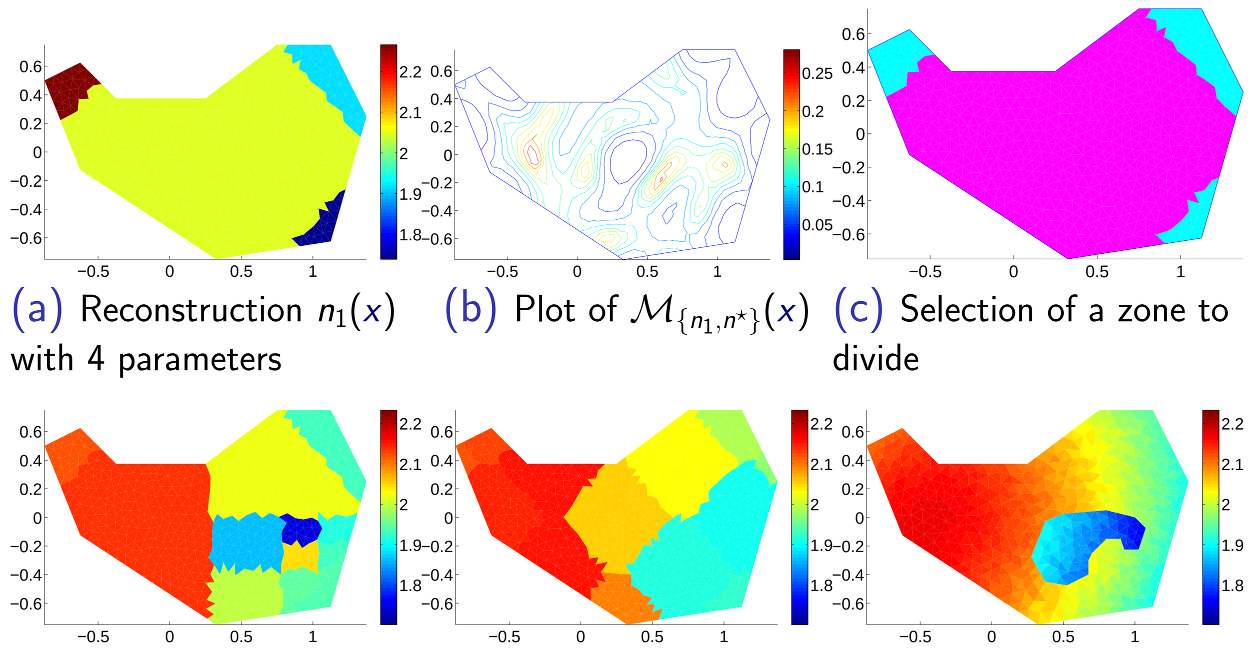
(a) Reference index $n(x)$ (b) Perturbed index $n^\star(x)$ (c) Level lines of $\mathcal{M}_{\{n,n^\star\}}$ (d) Selective reconstruction

Figure: Reconstruction of a perturbed index

Principle



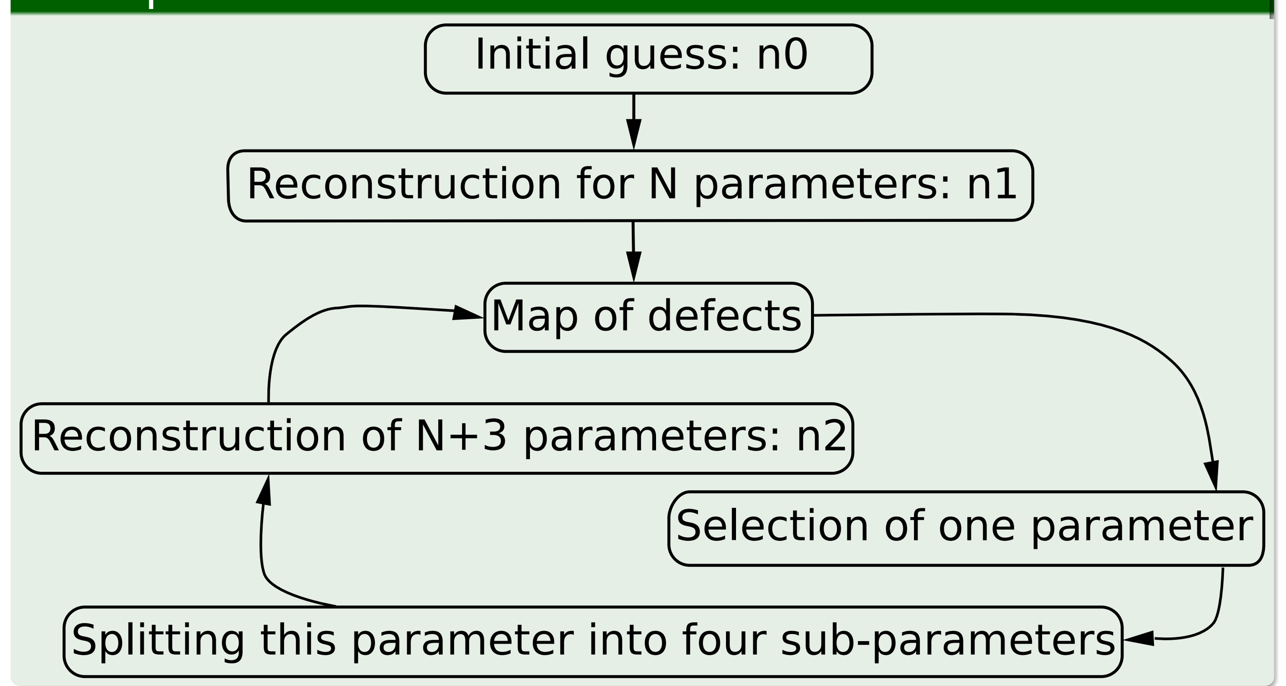
Application 2: adaptive refinement



(a) Reconstruction $n_1(x)$ with 4 parameters (b) Plot of $\mathcal{M}_{\{n_1,n^\star\}}(x)$ (c) Selection of a zone to divide (d) Reconstruction $n_4(x)$: 13 selected parameters (e) Reconstruction with 13 uniform parameters (f) Actual index $n^\star(x)$

Figure: Adaptive refinement

Principle



Uniqueness of the solution

Usual reconstruction of $n_\star(x)$:

$$\min J(n) := \|u^\infty_n - u^\infty_{n_\star}\|^2_{L^2(\Gamma_m)}$$

Theorem

- $n(x), n_\star(x) \in \mathbb{R}$
- $(n - n_\star)(x) > 0$ or < 0
- $\Gamma_e = \Gamma_m = S^{d-1}$

$$\mathcal{M}_{\{n,n_\star\}}(x) = 0 \iff n(x) = n_\star(x).$$

Perspective

Observation space: $L^2(O) \implies$ Domain decomposition through local convergence

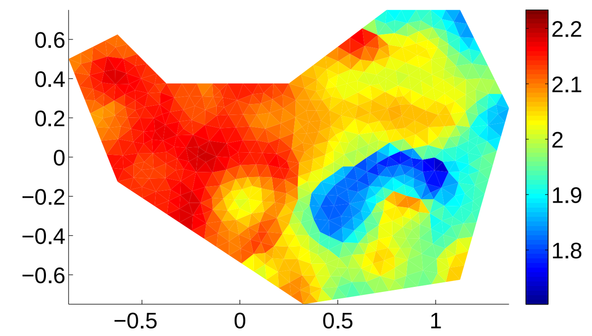


Figure: Reconstruction of $n_\star(x)$ by minimization of $J_{\mathcal{M}}(n) := \|\mathcal{M}_{\{n,n_\star\}}\|^2_{L^2(O)}$

Perspective

Minimization of a localization function $\implies$ better results through L^1-norm minimization

References

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