Locating the Support of Defects in An Inhomogeneous Medium from Far-Field Data

Yann Grisel[1,2], Pierre Mazet[1,2], Vincent Mouysset[1], Jean-Pierre Raymond[2]

[1] ONERA - Toulouse- France
[2] Université Toulouse 3, Paul Sabatier - France

r e t o u r s u r i n n o v a t i o n

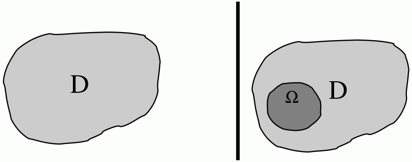


Figure: Object in initial state (left) - Objet with defects (right).

Problem

- Recover informations about defects in a known object from far-field measurements

Goal

- Numerically recover the shape of defects by a fast method

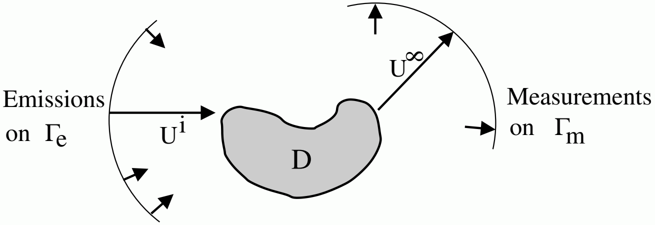


Figure: Acoustic scattering: plane-wave incidence directions and far-field measurements

- Helmholtz equation in an inhomogeneous medium with a plane wave source $u^i(\theta, x) := \exp(ik\theta \cdot x)$

$$\begin{cases} \Delta u_n^T(\theta,x) + k^2 n(x) u_n^T(\theta,x) = 0, & x \in \mathbb{R}^d, \\ u_n^T(\theta,x) = u_n^s(\theta,x) + \exp(ik\theta\cdot x), & \theta \in \Gamma_e,\ x \in \mathbb{R}^d, \\ \lim_{r\to\infty} r^{\frac{d-1}{2}}(\partial_r u_n^s - iku_n^s) = 0. \end{cases}$$

$$D := \mathrm{support}(n-1)$$

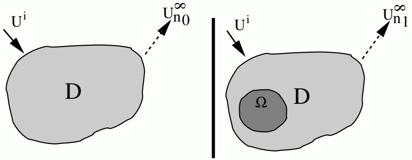


Figure: Object in initial state (left) - Objet with defects (right).

- Initial state refraction index: n_0
- Perturbed state refraction index: n_1
- Far-Field operator: $F_n g(\theta) = \left\langle g, \overline{u_n^\infty(\cdot,\theta)} \right\rangle_{L^2(\Gamma_e)}, \ \theta \in \Gamma_m.$
- Measurements operator: $\Delta_F := F_{n_1} - F_{n_0}$

Data:

- Initial state index n_0
- Post-perturbation measurements $u^\infty_{n_1}$

Goal: Reconstruct $\Omega := \mathrm{support}(n_1 - n_0)$ from measurements represented by the operator Δ_F

Reconstructing the shape of defects

Definition: k^2 is an "inhomogeneous transmission eigenvalue" if there exist a non-trivial solution (u, h) to the "inhomogeneous transmission problem" (ITP)

$$(ITP)\begin{cases}(\Delta + k^2 n_0)u = -k^2(n_1 - n_0)h \text{ in } \Omega,\\ (\Delta + k^2 n_1)h = 0 \text{ in } \Omega,\\ u = 0 \text{ on } \partial\Omega,\\ \partial_\nu u = 0 \text{ on } \partial\Omega.\end{cases}$$

Theorem (Reconstruction using a pseudo-inverse)

If k^2 is not an "inhomogeneous transmission eigenvalue", with real-valued indexes n_0 and n_1 and with full bi-static data, then $\forall x \in \mathbb{R}^d$ we have

$$\boxed{x \in \Omega \iff 0 < \mathscr{M}_{\{n_0,n_1\}}(x)}$$

where

$$\mathscr{M}_{\{n_0,n_1\}}(x) := \left\| W^{\ddagger}\,\overline{u^T_{n_0}(\cdot, x)} \right\|^{-2},$$

and $W^{\ddagger}$ is the pseudo-inverse of $W^{\frac{1}{2}}$ and W is built from F_{n_0} and F_{n_1}.

Sketch of proof

1 - Characterization of Ω by point sources: $\left(\Delta + k^2 n_0\right) G_{n_0}(x,\cdot) = -\delta_x$

1.
   - If $x \in \Omega$, let f_x be smooth and equal to $G_{n_0}(x,\cdot)$ outside some ball B such that $\{x\} \subset B \subset \Omega$, then
$$G_{n_0}^{\infty}(x,\theta) = f_x^{\infty}(\theta) = \left(V_{n_0}\left[\chi_\Omega(\Delta + k^2 n_0) f_x\right]\right)^{\infty}(\theta).$$
   - If $x \notin \Omega$, $V_{n_0}\chi_\Omega h$ is smooth outside Ω and $G_{n_0}(x,\cdot)$ has a singularity in $\{x\}$.
   - Conclusion: $x \in \Omega \iff G_{n_0}^{\infty}(x,\cdot) \in \mathcal{R}\left(V_{n_0}^{\infty}\chi_\Omega\right)$.
2. By the Lippmann-Schwinger equation
$$(V_n g)^{\infty} = \left(V_1\left(g + k^2(n-1)V_n g\right)\right)^{\infty},$$
we have the mixed reciprocity principle
$$G_n^{\infty}(x,\theta) = u_n^T(-\theta, x).$$
3. Thus, with the operator $T^\star : L^2(D) \to L^2(\Gamma_e)$, defined by
$$T^\star f(\theta) = \langle f, u_{n_0}^T(\theta,\cdot)\rangle_{L^2(D)},$$
for each $x \in \mathbb{R}^d$ we have
$$\boxed{x \in \Omega \iff \overline{u_{n_0}^T(\cdot, x)} \in \mathcal{R}\left(T^\star \chi_\Omega\right).}$$

2 - From point sources to measurements

1. Step 1: From defects to the range of some operator

$$x \in \Omega \iff \overline{u^{T}_{n_0}(\cdot, x)} \in \mathcal{R}\left(T^{\star}\chi_{\Omega}\right).$$

2. Step 2: [Nachman, 2007] From the range of an operator to a functional inequality

$$\varphi \in \mathcal{R}\left(L^{\star}\right) \iff \exists c > 0/\ \forall \Psi,\ |\langle \Psi,\ \varphi \rangle| \leqslant c\, \|L\Psi\|\,.$$

Thus, we have

$$\boxed{x \in \Omega \iff \exists c > 0/\ \forall \Psi,\ \left|\left\langle \Psi,\ \overline{u^{T}_{n_0}(\cdot, x)} \right\rangle\right| \leqslant c\, \|\chi_{\Omega} T\Psi\|\,.}$$

And $\|T\Psi\| = \langle T\Psi,\ T\Psi \rangle^{\frac{1}{2}} = \langle T^{\star} T\Psi,\ \Psi \rangle^{\frac{1}{2}}$.

Question: is $\|\chi_{\Omega} T\Psi\|$ comparable to $\langle \Delta_F \Psi,\ \Psi \rangle^{\frac{1}{2}}$?

3 - Factorization of the measurements operator

1. The operator $\Delta_F := F_{n_1} - F_{n_0}$ has a factorization of the form

$$\Delta_F = V^{\infty}_{n_0} A T.$$

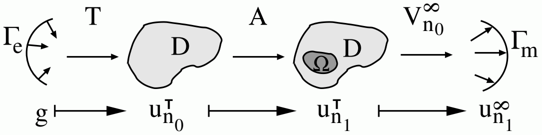


Figure: Factorization of the measurements operator.

2. With the scattering operator $S = (Id + 2ik\,|\gamma|^2\, F_{n_0})$, we have

$$V^{\infty}_{n_0} = S T^{\star}.$$

3. Thus, the product of operators $S^{\star}\Delta_F$ has a factorization of the form

$$\boxed{S\Delta_F = T^{\star} A T.}$$

4. The operator A is an isomorphism from $L^2(\Omega)$ into itself and is coercive.

4 - Factorization of a positive and self-adjoint operator

$$\text{Notation: } |L| = (L^\star L)^{\frac{1}{2}}.$$

1. Based on [Kirsch, 2002], the operator W, built from the measurements and defined by

$$W = |S\Delta_F - (S\Delta_F)^\star| + |S\Delta_F + (S\Delta_F)^\star|,$$

has a factorization of the form

$$W = T^\star B T,$$

where B is coercive, positive and self-adjoint .

2. Thus, we obtain
   - The functional $\langle W\Psi, \ \Psi\rangle^{\frac{1}{2}}$ is comparable to $\|\chi_\Omega T\Psi\|$
   - The operator W is positive and self-adjoint.

5 - Conclusion

$$
\begin{aligned}
x \in \Omega &\iff \overline{u^T_{n_0}(\cdot, x)} \in \mathcal{R}\left(T^\star \chi_\Omega\right) && \text{(step 1)}\\
&\iff \forall \Psi,\ \left|\langle \Psi,\ \overline{u^T_{n_0}(\cdot, x)}\rangle\right| \leqslant c\, \|\chi_\Omega T\Psi\| && \text{(step 2)}\\
&\iff \forall \Psi,\ \left|\langle \Psi,\ \overline{u^T_{n_0}(\cdot, x)}\rangle\right| \leqslant c\langle S^\star \Delta_F \Psi,\ \Psi\rangle^{\frac{1}{2}} && \text{(factorization 1)}\\
&\iff \forall \Psi,\ \left|\langle \Psi,\ \overline{u^T_{n_0}(\cdot, x)}\rangle\right| \leqslant c\langle W\Psi,\ \Psi\rangle^{\frac{1}{2}} && \text{(factorization 2)}\\
&\iff 0 < \inf\left\{\langle W\Psi,\ \Psi\rangle^{\frac{1}{2}},\quad \left\langle \Psi,\ \overline{u^T_{n_0}(\cdot, x)}\right\rangle = 1\right\}
\end{aligned}
$$

W positive and self-adjoint: this minimization problem has an explicit solution

▶ Characterization by $\mathscr{M}_{\{n_0, n_1\}}(x) > 0$

Notation: $|L| = (L^\star L)^{\frac{1}{2}}$.

Lemma

The operator $|\Delta_F| := |F_{n_1} - F_{n_0}|$ *defined by*

$$|\Delta_F| = (\Delta_F^\star \Delta_F)^{\frac{1}{2}},$$

is positive, self adjoint and satisfies

$$|\Delta_F|^2 = T^\star \left|V_{n_0}^\infty A\right|^2 T.$$

Conjecture: One can replace the operator W by the operator $|\Delta_F|$ in the characterisations of Ω by the function $\mathscr{M}_{\{n_0,n_1\}}$.

Numerical examples

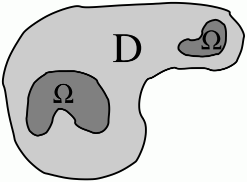
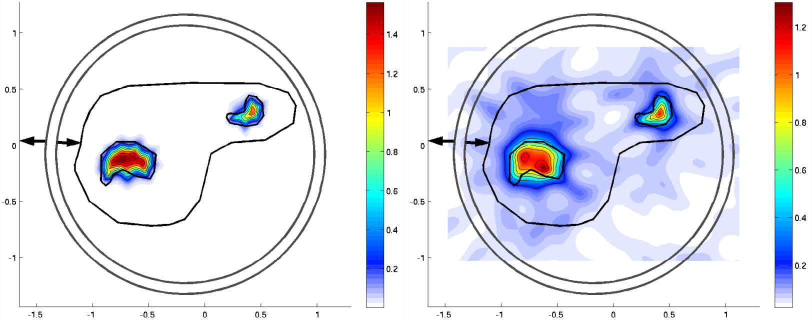


Figure: Shapes of the studied object an his defects

- Object size ≈ wavelength
- Defects size ≈ wavelength/5

Validation on a non-trivial example



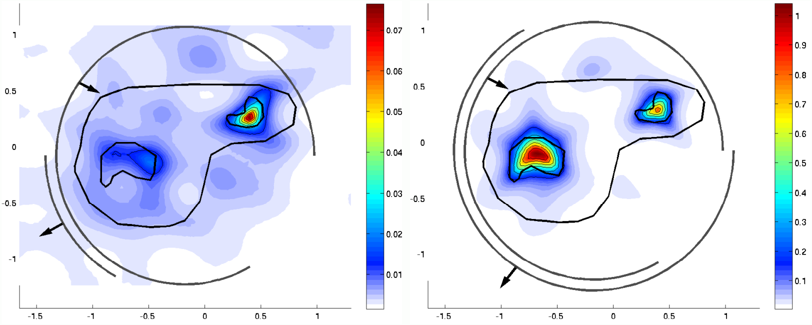
(a) 0% noise added to the measurements (b) 10% noise added to the measurements

Figure: Map of values for $\mathscr{M}_{\{\boldsymbol{n_0},\boldsymbol{n_1}\}}(z_i)$ with the operator W

- $\Gamma_e = \Gamma_m = [0\pi, 2\pi]$ with 35 directions
- $n_0(x) \in [1.33, 1.41]$ in D
- $n_1(x) \in [1.95, 2.01]$ in Ω

ONERA

THE FRENCH AEROSPACE LAB



(a) $\Gamma_m = [0.667\pi,\ 2\pi]$, 40 measurements (b) $\Gamma_m = [1\pi,\ 1.33\pi]$, 40 measurements

Figure: Map of values for $\mathscr{M}_{\{n_0,n_1\}}(z_i)$ with the operator $|\Delta_F|$

- $\Gamma_e = [0\pi,\ 1.67\pi]$ with 35 incidence directions
- 2% noise added to the measurements
- $n_0(x) \in [1.53,\ 1.63] + [0.07,\ 0.12]i$ in D
- $n_1(x) \in [2.26,\ 2.33] + [0.48,\ 0.52]i$ in Ω

Achievements

- The Factorization Method was applied to the reconstruction of the shape of defects in an inhomogeneous medium from far-field measurements
- A possible extension of our results was numerically confirmed

Prospects

- Proving the extension of our methods to absorbing objects and with measurements directions different from the incidence directions
- Adding the reconstruction of the near-field test-functions from the measurements $\to$ Motion detection
- using these informations as an error indicator in an iterative index reconstruction scheme

Thank you for your attention